King Fahd University of Petroleum and Minerals

College of Computer Science and Engineering
Information and Computer Science Department

ICS 353: Design and Analysis of Algorithms
Spring 2006-2007
Major Exam 1, Monday March 19, 2007.

Name: Possible Solutions

ID#:

Instructions:

1. This exam consists of 8 pages, including this page, containing 5 questions.
2. You have to answer all 5 questions.
3. The exam is closed book and closed notes. No calculators or any helping aides are allowed. Make sure you turn off your mobile phone and keep it in your pocket if you have one.
4. The questions are equally weighed.
5. The maximum number of points for this exam is 150.
6. You have exactly 90 minutes to finish the exam.
7. Make sure your answers are readable.
8. If there is no space on the front of the page, feel free to use the back of the page. Make sure you indicate this in order for me not to miss grading it.

Question	Max Points	Points
1	30	
2	30	
3	30	
4	30	
5	30	
Total	150	

Some Useful Formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1} \text{ where } a \neq 1, \quad \log\frac{a}{b} = \log a - \log b, \quad \log_b a = \frac{\ln a}{\ln b}$$

1. (30 points) Answer the following:
   a. (15 points) Prove that $\log n = o(n^{\varepsilon})\ \forall \varepsilon > 0$

5 —— $\lim\limits_{n \to \infty} \frac{\log n}{n^{\varepsilon}}$

5 —— $= \lim\limits_{n \to \infty} \frac{1}{n \ln 2 \cdot \varepsilon n^{\varepsilon - 1}}$ (Using L'Hopital's rule)

3 —— $= c \cdot \lim\limits_{n \to \infty} \frac{1}{n^{\varepsilon}}$ $\left[c = \frac{1}{\varepsilon \cdot \ln 2}, \text{a constant} \right]$

$= c \cdot \frac{1}{\infty} = \emptyset.$

2 —— $\therefore \log n = o(n^{\varepsilon})\quad \forall \varepsilon > 0.$

   b. (15 points) Given two functions $f(n)$ and $g(n)$, if $\log f(n) = \Theta(\log g(n))$, does this imply that $f(n) = \Theta(g(n))$? If yes, prove it, and if no, give a counter example to show that the above statement is not true.

5 —— No.

$f(n) = n^2$, $g(n) = n^3$

10 —— $\log f(n) = 2n$ $\log g(n) = 3n$

$\log f(n) = \Theta(\log g(n))$

However, $f(n) \neq \Theta(g(n))$.

2. (30 points): Consider the following algorithm

```
1. count ← 0
2. for i ← 1 to ⌊log n⌋
3.    for j ← 1 to i
4.       for k ←1 to j
5.          count ← count + 1
6.       end for
7.    end for
8. end for
```

a. (15 points) Find the value of *count* by determining the number of times step 5 gets executed in this algorithm?
b. (5 points) What is the time complexity of the algorithm?
c. (10 points) Do you expect the average case time complexity to be asymptotically different from the best case time complexity of this algorithm? Justify your answer.

a. $$\sum_{i=1}^{\lfloor \log n \rfloor} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 \quad \text{———— } 7$$

$$= \sum_{i=1}^{\lfloor \log n \rfloor} \sum_{j=1}^{i} j \quad \text{———— } 9$$

$$= \sum_{i=1}^{\lfloor \log n \rfloor} \frac{i(i+1)}{2} \quad \text{———— } 11$$

15 —

$$= \frac{1}{2} \left(\sum_{i=1}^{\lfloor \log n \rfloor} i^2 + \sum_{i=1}^{\lfloor \log n \rfloor} i \right)$$

$$= \frac{1}{2} \frac{\left[\lfloor \log n \rfloor (\lfloor \log n \rfloor + 1)(2\lfloor \log n \rfloor + 1)\right]}{6} + \frac{1}{2} \frac{\lfloor \log n \rfloor (\lfloor \log n \rfloor + 1)}{2}.$$

b. $\Theta(\log^3 n)$. —— 5

c. No. The algorithm is exact, in the sense that the number of operations does not depend on the input values, only on the input size.

4

6

3. (30 points) Consider the max-heap data structure. Given as input a max-heap array with n elements, where $n > 4$:
   a. (20 points) Develop an efficient algorithm that runs in $o(n)$, which returns the 4^{th} largest element in the array. You may use any algorithm we discussed related to heaps without rewriting them.
   b. (10 points) Analyze the time complexity of the algorithm you developed in part a.

a. Algorithm Fourth (Heap H)
   1. for $i \leftarrow 1$ to 3
   2. DeleteMax (H);
   3. return FindMax (H);

Those who used heapsort : Ø pts
Those who checked 3^{rd} level : 5 pts
Algorithm not rightly/well written : −10 or −5

b. Each deleteMax will cost $O(\log n)$.
∴ lines 1 & 2 run in $\leq 3 \cdot (c \cdot \log n)$ which is $\Theta(\log n)$
constant

line 3 runs in $\Theta(1)$

∴ the algorithm runs in $\Theta(\log n)$, which is $o(n)$.

Did not conclude with $\Theta()$ or $O()$ notation : −2pts

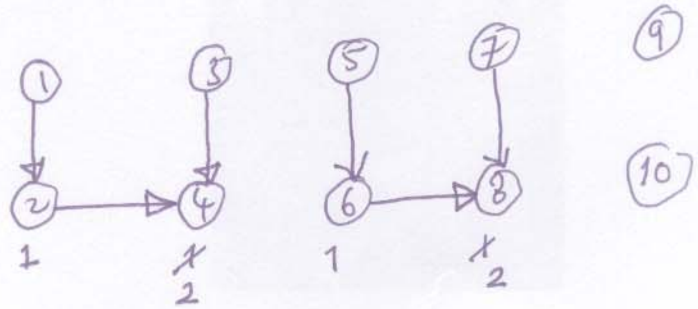
4. (30 points) Consider the ten-singleton disjoint sets

$$\{1\}\ \{2\}\ \{3\}\ \{4\}\ \{5\}\ \{6\}\ \{7\}\ \{8\}\ \{9\}\ \{10\}$$

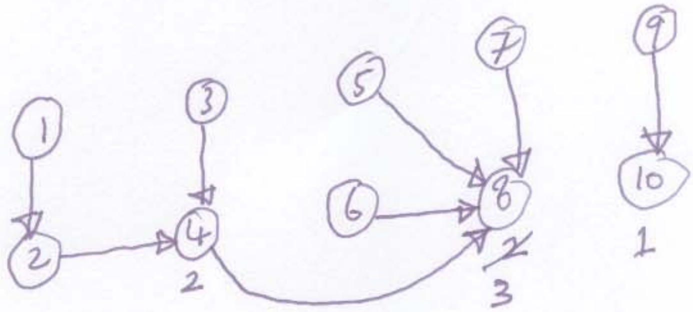
a. (20 points) Using the union by rank and path compression heuristics, show the result of performing the following sequence of unions: Union(1, 2) , Union(3, 4) , Union(5, 6) , Union(7, 8) , Union(1, 4) , Union(5, 7) , Union(4, 5) , Union(9, 10) , Union(3,7) , Union(1, 9)

b. (10 points) What is the time complexity of the union-find algorithms, when we use union by rank without path compression and when we use union by rank with path compression? Do they belong to the same time complexity class?

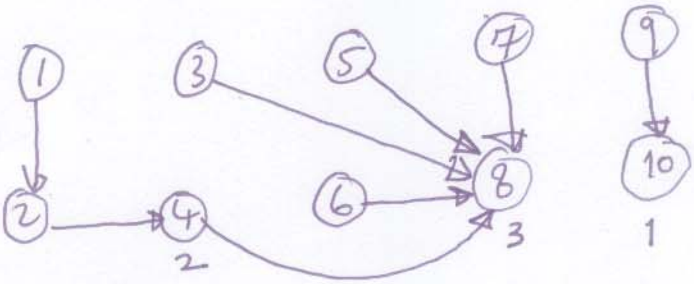
a.



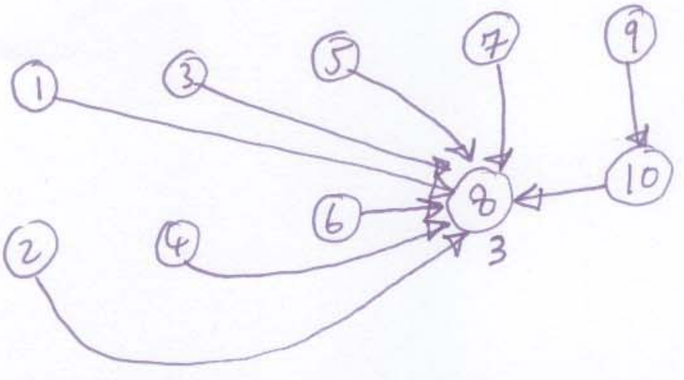
— 8



— 12



— 15



— 20

For n nodes with m union/find operations.

b. i. Union by rank without path compression:

~~For A~~

4 — $T(m,n) = O(m \log n)$

ii. Union by rank with path compression:

4 — $T(m,n) = O(m \log^* n)$

where $\log^* n = \{\text{smallest } i \text{ s.t. } \underbrace{\log \log \ldots \log}_{i \text{ times}} n \le 1\}$.

2 — No. They belong to different time complexities.

A wrong answer to b.i or b.ii will lead to losing the 2 pts

5. (30 points) Find asymptotically tight bounds, in terms of Θ, for the following functions:

a. (15 points) $f(n) = \begin{cases} 20 & n = 1 \\ 2f(n-1) & n > 1 \end{cases}$

b. (15 points) $g(n) = \begin{cases} 5 & n = 0 \\ 10 & n = 1 \\ 4(g(n-1) - g(n-2)) & n \geq 2 \end{cases}$

a. $f(n) = 2f(n-1)$

$= 2 \cdot (2f(n-2))$

$= 2^2 f(n-2)$

$= 2^3 f(n-3)$

$\vdots$

$= 2^{n-1} f(n-(n-1)) = 2^{n-1} f(1)$ ——— 10

$= 20 \cdot 2^{n-1}$

$= 10 \cdot 2^n$

$\therefore f(n) = \Theta(2^n)$. ——— 5

b. This is a homogeneous equation of the form:

2 ——— $g(n) = a_1 g(n-1) + a_2 g(n-2)$, where $a_1 = 4$, $a_2 = -4$

$\therefore$ Characteristic Equation:

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

3 ——— $x = 2$.

5 ——— $\therefore g(n) = c_1 2^n + c_2 n 2^n$

$g(0) = c_1 = 5$

$g(1) = 2c_1 + 2c_2 = 10$

$c_1 + c_2 = 5$

$c_2 = 0$

3 ——— $\therefore g(n) = 5 \cdot 2^n$

2 ——— $\therefore g(n) = \Theta(2^n)$.